Chapter 4

Optimization over a convex set

Consider an optimization problem over a *convex feasible set S*:

$$\begin{array}{ll} min & f(x) \\ & x \in S. \end{array} \qquad (P-CONV)$$

Theorem 4.1 (Feasible directions convex set) *Let $S \subseteq \Re^n$ be a convex set and $\bar{x} \in S$. The direction $d = x - \bar{x}$, for any $x \in S$ such that $x \neq \bar{x}$, is a feasible direction for S in $\bar{x}$.*

Proof. For all $x \in S$ with $x \neq \bar{x}$, by convexity of S, we have that $(1-\beta)\bar{x}+\beta x \in S$ for all $\beta \in [0,1]$ and hence $\bar{x}+\beta(x-\bar{x}) \in S$ for all $\beta \in [0,1]$. hence the direction $d = x - \bar{x}$ is feasible for S in $\bar{x}$.

On the other hand, any feasible direction $d \neq 0$ for S in $\bar{x}$ can be written as $d = t(x-\bar{x})$ for some $x \in S$ and $t \in \mathbb{R}_+$.

Using Theorem 3.14we get:

Theorem 4.2 (First order optimality condition over a convex set) *Let $x^* \in S$ be a local minimizer of problem (P-CONV) and assume that f is continuously differentiable over$\Re^n$. Then it holds:*

$$\nabla f(x^*)^T(x-x^*) \geq 0, \quad \text{for all} \quad x \in S. \tag{4.1}$$

When also the function f is convex, we have the following.

Theorem 4.3 (Necessary and sufficient condition for convex problem) *Let S be a convex subset of R^n and f is convex function continuously differentiable over$\Re^n$. The point $x^* \in S$ is a global minimizer of problem (P-CONV) if and only if (P-CONV) se e solo se*

$$\nabla f(x^*)^T(x-x^*) \geq 0, \quad \text{for all} \quad x \in S. \tag{4.2}$$

Using second order characterization of feasible directions given by theorem 3.12 we get

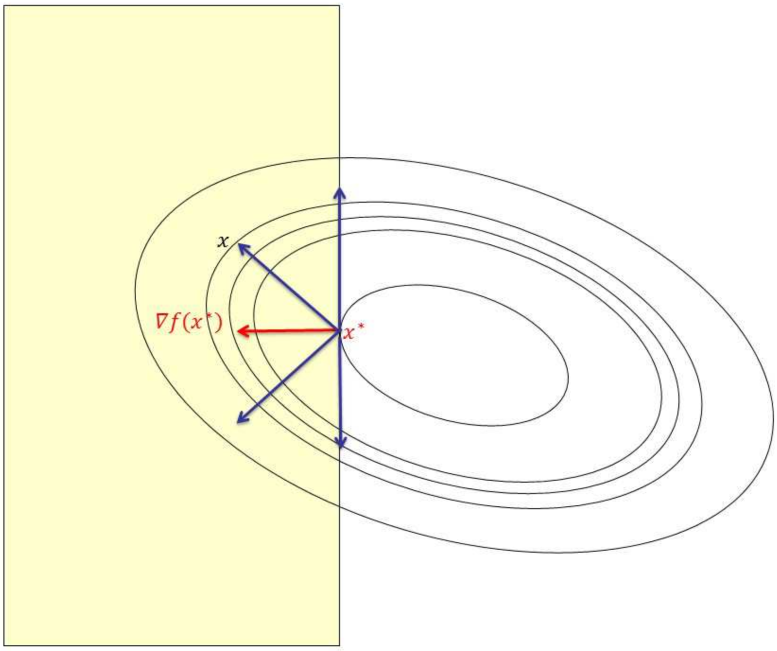

Figure 4.1: Rappresentazione grafica delle condizioni di ottimo su insieme convesso.

Theorem 4.4 (Second order necessary conditions over a convex set) *Let $x^* \in S$ be a local minimizer of problem (P-CONV) and assume that f is twice continuously differentiable over an open set containing S. Then it holds:*

$$(x-x^*)^T \nabla^2 f(x^*)(x-x^*) \geq 0, \quad \text{for all} \quad x \in S \quad \text{such that} \quad \nabla f(x^*)'(x-x^*) = 0. \tag{4.3}$$

Example 4.5 let's consider the example

$$\begin{array}{ll} \min & -x_1^2 - x_2^2 \\ & -1 \leq x_1 \leq 1 \\ & -1 \leq x_2 \leq 1 \end{array}$$

The feasible set is convex. The first order optimality condition in a feasible point $\bar{x}$ are written as follows.

$$\nabla f = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix}$$

We have

$$-2\bar{x}_1(x_1 - \bar{x}_1) - 2\bar{x}_2(x_2 - \bar{x}_2) \geq 0 \quad \text{per ogni} \quad -1 \leq x_1 \leq 1 \quad -1 \leq x_2 \leq 1.$$

Let us check the optimality condition above in the point $x = (1,\ 1)^T$.

$$-2(x_1 - 1) - 2(x_2 - 1) \geq 0 \ \text{ for all } \ -1 \leq x_1 \leq 1 \quad -1 \leq x_2 \leq 1.$$

The terms $x_1 - 1 \leq 0$, $x_2 - 1 \leq 0$ for any $x \in S$, hence the condition is satisfied.

4.6 Algorithmic use: conditional gradient method

Consider the problem

$$\min_{x\in S} f(x) \tag{4.4}$$

with $f : R^n \to R$ and S convex and assume

Assumption 4.7 *The function $f : R^n \to R$ is continuosuly differentiable and S is compact.*

which gurantees existence of a solution. Algorithms for the solution of problem (4.4)find point in the target set

$$\Omega := \{\omega \in R^n : \nabla f(\omega)^T (x-\omega) \geq 0, \text{per ogni } x \in S\}.$$

A possible scheme is

Optimization algorithm over a convex set S

1. Fixed a *starting point* $x^0 \in R^n$ and set $k = 0$.
2. If $x^k \in \Omega$ stop.
3. Evaluate a *feasible and descent direction* $d^k \in R^n$.
4. Evaluate a *stepsize* $\alpha^k \in R$ along d^k such that $x^k + \alpha^k d^k \in S$;
5. Find a new point $x^{k+1} = x^k + \alpha^k d^k$. Set $k = k+1$ and go to Step 2.

Choice of the direction and of the stepsize.
At a point x^k, consider the onstrained problems

$$\min_{x\in S} \nabla f(x^k)^T (x-x^k). \tag{4.5}$$

The objective function is linear and S is convex and compact so that problem (4.5) is convex and it admits a global solution x^{k*}. By definition of minimizer it results

$$\nabla f(x^k)^T (x-x^k) \geq \nabla f(x^k)^T (x^{k*}-x^k) \quad \forall\, x \in S$$

If $\nabla f(x^k)^T d^k = \nabla f(x^k)^T (x^{k*}-x^k) \geq 0$ then x^k satisfies the first order optimality condition and hence it satsfies the condition at Step 2. If otherwise $\nabla f(x^k)^T (x^{k*}-x^k) < 0$ the direction $d^k = x^{k*}-x^k$ is a descent direction. The stepsize $\alpha^k > 0$ can be determined by approximate or exact line search along d^k such that $x^{k+1} \in S$ and $f(x^{k+1}) < f(x^k)$. This algorithm is know as *Frank-Wolfe method* or *Conditional gradient method*.