Chapter 1

Optimization problems

An optimization problem consists in maximizing or minimizing some function relative to some set, representing a range of choices available in a certain situation. The function allows comparison of the different choices for determining which might be best.

More formally we define the optimization problem as

$$\begin{array}{l} \text{optimize} \quad f(x) \\ x \in S \end{array} \tag{1.1}$$

where *optimize* stands for min or max $f : R^n \to R$ denotes the *objective function*, that we assume throughout at least continuously differentiable, and $S \subseteq R^n$ is the *feasible set*, namely the set of all admissible choices for x.

In the following we will refer to *minimization* problems. Indeed the optimal solution of a maximization problem

$$\begin{array}{ll} max & f(x) \\ & x \in S \end{array}$$

coincide with the optimal solutions of the minimization problem

$$\begin{array}{ll} min & -f(x) \\ & x \in S \end{array}$$

and we have: $\max_{x \in S} f(x) = -\min_{x \in S}(-f(x))$.

The feasible set S is a subset of $\mathbb{R}^n$ and hence $x = (x_1, x_2, \ldots, x_n)^T$ is the vector of variables of dimension n and f is a function of n real values $f(x_1, x_2, \ldots, x_n)$.

1.1 Preliminary definitions

Definition 1.2 (Unfeasible problem) *An optimization problem is said to be* unfeasible *if $S = \emptyset$, namely if there are no admissible choices.*

Definition 1.3 (Unbounded Problem) *A minimization (maximization) problems is* unbounded below (above) *if for any value $M > 0$ a point $x \in S$ exists such that $f(x) < -M$.*

An example of unbounded problem is $\min_S f(x) = x^3$ with $S = \{x : x \leq 2\}$. Indeed for $x \to -\infty$, the function $f \to -\infty$ too. Please note that the same function may admits a minimizer on a different feasible set. For example consider $S = \{x : x \geq 0\}$,ad the problem is no more unbounded.

Definition 1.4 (Global minimizer or optimal solution) *A point x^* is a* global minimizer *if*

$$f(x^*) \leq f(x) \text{ for all } x \in S.$$

When S is an open set (in particular when $S := \mathbb{R}^n$) we refer to unconstrained minimizer; otherwise we refer to a constrained minimizer.

An optimization problems admits a solution if a global minimizer $x^\star \in S$ exists. The corresponding value $f(x^\star)$ is called *optimal value*.

Per esempio, se si pone $f = x^2$ e $S = \mathbb{R}$, l'ottimo è l'origine, e il corrispondente valore ottimo è zero. Se si prende $S = \{x : x \geq 2\}$, l'ottimo è 2 e il valore ottimo 4.

Generally, we look for a global minimizer x^* of f, namely a point where the function attains its least value. The formal definition is The global minimizer can be difficult to find, as it will be clearer in the following. Indeed most algorithms are able to find only a local minimizer, which is a point that achieves the smallest value of f only in its neighborhood. Formally, we say:

Definition 1.5 (Local minimizer) *A point $\bar{x} \in S$ is a* local minimizer *if there exists a neighborhood $\mathcal{N}(\bar{x}, \rho)$ of $\bar{x}$ such that*

$$f(\bar{x}) \leq f(x) \text{ for all } x \in \mathcal{N} \cap S.$$

A point $\bar{x} \in S$ is a strict local minimizer *if there is a neighborhood $\mathcal{N}(\bar{x}, \rho)$ of $\bar{x}$ such that*

$$f(\bar{x}) < f(x) \text{ for all } x \in \mathcal{N} \cap S \text{ with } x \neq \bar{x}.$$

Of course any global minimizer is also a local one, but not vice versa.

It can also happen that the objective function is bounded below on S, namely that:

$$\inf_{x \in S} f(x) > -\infty,$$

but *there exists no global minimizer* of f on S.
"Solving" an optimization problem meams,

- verify if the feasible set in not empty or conclude that feasible solutions do not exist;

- verify if an optimal solution exists or prove that the problem do not admit solutions;
- find an optimal solution

1.6 Class of problems

- **Continuous Optimization**

  The variables x can take values in $\mathbb{R}^n$ (continuous values); we can further distinguish in

  - *constrained problems* if $S \subset \mathbb{R}^n$
  - *unconstrained problems* if $S = \mathbb{R}^n$.

- **Discrete Optimization.**

  The variables x can take values only on a finite set; we can further distinguish in:

  - *integer programming* if $S \subseteq \mathbf{Z}^n$
  - *boolean optimization* if $S \subseteq \{0,1\}^n$.

- **Mixed problems.**

  Some of the variables are continuous and soe are discrete.

The feasible set is usually expressed by a finite number of equality or inequality relations.

Formally consider the functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\dots,m$ a feasible set defined by inequality constraints is

$$S = \{x \in \mathbb{R}^n \ | \ g_1(x) \le 0, \ g_2(x) \le 0, \ \dots \ , \ g_m(x) \le 0\}.$$

Each inequality $g_i(x) \le 0$ is called *constraint* and the feasible set is made up of points solving the system of nonlinear inequalitiees

$$\begin{array}{rcl} g_1(x) & \le & 0 \\ g_2(x) & \le & 0 \\ g_3(x) & \le & 0 \\ & \vdots & \\ g_m(x) & \le & 0 \end{array}$$

Please note that any constraint of the form $g(x) \ge 0$ can be reported in the form above by simple multiplying by minus one, namely $-g(x) \le 0$. Further an equality constraint $h(x) = 0$ can also be transformed into two inequality constraints as $h(x) \leq 0$ e $-h(x) \leq 0$. However in some cases equality constraints can be treated explicity, so that we consider a generic problem of the form

$$\begin{array}{ll} \min & f(x) \\ & g_i(x) \leq 0, \;\; i = 1, \dots, m \\ & h_j(x) = 0, \;\; j = 1, \dots, p. \end{array} \tag{1.2}$$

or in more compact form as

$$\begin{array}{ll} \min & f(x) \\ \\ & g(x) \leq 0, \\ & h(x) = 0 \end{array} \tag{1.3}$$

where $g : R^n \to R^m$ $h : R^n \to R^p$.

The optimization problem is called *linear program* or Linear programming problem (LP) if the objective and constraint functions $f, g_1, \dots, g_m, h_1 \dots . h_p$ are linear, i.e., satisfy the condition

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $x, y \in \mathbb{R}^n$ and given $\alpha, \beta \in \mathbb{R}$.

In general we write an LP in the form

$$\begin{array}{ll} \min & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ \\ & a_{i1} x_1 + \quad \dots \quad + a_{in} x_n \approx b_i \end{array} \tag{1.4}$$

where $\approx$ stands for $\geq$, $\leq$ or $=$.

If the optimization problem is not linear (namely at least one of the functions f, g_i, h_j is not linear), it is called a *nonlinear program* or Nonlinear programming problem (NLP)

Some example for Mathematical programming problem follow.

Esempio 1 *Consider the minimization of the function in two variables* $f(x_1, x_2) = 2x_1 + x_2$ *under the constraints* $2x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0$. *The optimization problems is*

$$\begin{array}{ll} \min & 2x_1 + x_2 \\ & x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

which is in the form (1.2) *where* $g_1(x_1, x_2) = x_1 + x_2 - 1$, $g_2(x_1, x_2) = -x_1$, $g_3(x_1, x_2) = -x_2$. *This is a Linear programming problem.*

Esempio 2 *Let us consider the minimization of the function* $f(x_1, x_2) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2$ *subject to the constraints* $x_1 + x_2 \geq 1$, $x_1 \leq 1$, $x_2 \leq 1$. *We get the nonlinearprogranning problem*

$$\begin{array}{ll} \min & -(x_1 - \frac{1}{2})^2 - (x_2 - \frac{1}{2})^2 \\ & x_1 + x_2 \geq 1 \\ & x_1 \leq 1 \\ & x_2 \leq 1 \end{array}$$